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THE ABSOLUTE MINIMUM IN THE PROBLEM OF THE SURFACE OF REVOLUTION OF MINIMUM AREA

BY MARY EMILY SINCLAIR

Introduction. It is well known that the problem of finding a relative minimum of the integral

$$I = \int y\sqrt{x'^2 + y'^2}\, dt$$

with respect to all ordinary * curves which can be drawn in the region

$$R:\ y \geqq 0$$

from a given point A_0 to another given point A_1, has either one or two solutions, according to the position of the two points A_0 and A_1. When the two points are sufficiently far apart there is only one solution, the Goldschmidt discontinuous solution,† consisting of the ordinates of A_0 and A_1 together with the segment of the x-axis between them. If A_0 and A_1 are near to each other there are two solutions, the discontinuous solution just described, and, besides, an arc of a catenary with the x-axis as directrix.

These known results would immediately furnish also the solution of the problem of the *absolute* minimum of the above integral, *if we knew a priori the existence of an absolute minimum.* But such an existence proof ‡ has never been given, and the problem of the absolute minimum of the integral is, as far as I know, still unsolved. To fill this gap in the theory of the surface of revolution of minimum area is the object of the following pages.

We make use of the following lemma, due essentially to Todhunter: §

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* In the terminology of Bolza: *Lectures on the Calculus of Variations*, p. 117.

† Goldschmidt, *Göttingen Prize Essay*, 1831. Todhunter, *Researches in the Calculus of Variations*, p. 60. Bolza, loc. cit., p. 153.

‡ The assumptions under which Hilbert's existence theorem has been proved by Bolza (loc. cit. p. 247) are not satisfied for the region $R:\ y \geqq 0$.

§ Todhunter: loc. cit., p. 60, §§64, 65.

LEMMA: *If M_0A_0 is the ordinate of A_0, and if A_0L_0 is the arc of any ordinary curve not identical with A_0M_0, and such that arc $A_0L_0 = |A_0M_0|$, then,*

$$I(A_0L_0) > I(A_0M_0)^*$$

Let P and P' be any two points on A_0M_0 and A_0L_0 respectively, such that

$$|A_0P| = \text{arc } A_0P',$$

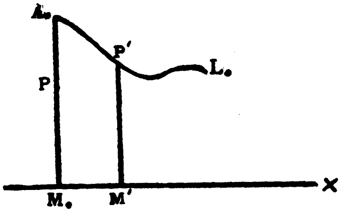

FIG. 1.

and let $M'P'$ be the ordinate of P'. When A_0P' coincides with A_0P, we have $|M'P'| = |M_0P|$. When A_0P' is different from A_0P, it follows that $|M'P'| > |M_0P|$ since a normal is the shortest distance from a point to a line.

Since L_0 does not coincide with M_0, there must be points P' of A_0L_0 for which A_0P' does not coincide with A_0P, the corresponding segment of A_0M_0. For such points

$$|M'P'|ds > |M_0P|ds,$$

and hence,

$$\int |M'P'|ds > |M_0P|\, ds,$$

or

$$I(A_0L_0) > I(A_0M_0).$$

This leads to the following theorem: *The Goldschmidt solution consisting of the path $A_0M_0M_1A_1$, furnishes a value of I less than that furnished by any other ordinary curve $\mathfrak{C}$ in $R(y \geqq 0)$ joining A_0 and A_1, whose length is equal to or greater than $|A_0M_0| + |A_1M_1|$.*

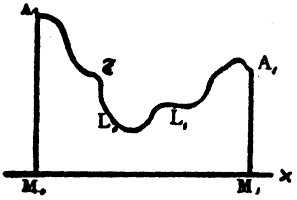

FIG. 2.

For let the arc A_0L_0 of $\mathfrak{C}$ equal $|A_0M_0|$, and let the arc A_1L_1 of $\mathfrak{C}$ equal $|A_1M_1|$. When A_0L_0 coincides with A_0M_0, and A_1L_1 with A_1M_1,

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* $I(AB) = I(BA)$.

then
$$I_{\mathfrak{C}}(A_0L_0) = I(A_0M_0),$$
and
$$I_{\mathfrak{C}}(A_1L_1) = I(A_1M_1).$$

But $I(M_0M_1) = 0$ and $I_{\mathfrak{C}}(L_0L_1) > 0$, since $\mathfrak{C}$ cannot coincide throughout with $A_0M_0\,M_1A_1$. Hence, $I_{\mathfrak{C}}(A_0A_1) > I(A_0M_0M_1A_1)$.

When A_0L_0 and A_1L_1 do not both coincide with A_0M_0 and A_1M_1, respectively, then, by the lemma,
$$I_{\mathfrak{C}}(A_0L_0) + I_{\mathfrak{C}}(A_1L_1) > I(A_0M_0) + I(A_1M_1).$$
Hence
$$I_{\mathfrak{C}}(A_0A_1) \geqq I_{\mathfrak{C}}(A_0L_0) + I_{\mathfrak{C}}(A_1L_1) > I(A_0M_0M_1A_1).$$

Hence the theorem is always true.

COR. $I_{\mathfrak{C}}(AB) > I(A_0M_0M_1A_1)$ *if* $\mathfrak{C}$ *is any rectifiable curve whose length is greater than* $|A_0M_0| + |A_1M_1|$. For, let arc $A_0L_0 = |A_0M_0|$ and arc $A_1L_1 = |A_1M_1|$, let N_0 and N_1 be two points of $\mathfrak{C}$ in the order $L_0N_0N_1L_1$. Consider any partition π, and a polygon P_π inscribed in A_0N_0. Then, if the partition is sufficiently fine, the length of P_π is greater than the arc A_0L_0. According to the Todhunter lemma,
$$I_{P_\pi}(A_0N_0) \geqq I(A_0M_0),^{*}$$
and, passing to the limit,†
$$I^*_{\mathfrak{C}}(A_0N_0) \geqq I(A_0M_0).$$
Similarly,
$$I^*_{\mathfrak{C}}(A_1N_1) \geqq I(A_1M_1).$$
Moreover,
$$I^*_{\mathfrak{C}}(N_0N_1) > 0.$$
Hence,
$$I^*_{\mathfrak{C}}(A_0A_1) > I(A_0M_0M_1A_1).$$

We may now apply the above results to our problem. We consider two cases.

Case I. $|A_0A_1| \geqq y_0 + y_1$. Then the length of every ordinary curve joining A_0 and A_1 is equal to or greater than $y_0 + y_1$. By the above theorem,

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[*] We use here the notation and definition of Bolza, loc. cit., p. 157, b).

[†] The integral $I = \int y\sqrt{x'^2 + y'^2}\,dt$ has not in general a well defined meaning unless the curve over which it is taken is represented by functions $x = x(t)$, $y = y(t)$ which have derivatives. The integrals $I_{P_\pi}(A_0N_0)$ have a limit, however, whenever the polygons over which they are taken are inscribed in a rectifiable curve. This limit is a generalization of the value of I denoted by I^*. See Bolza, loc. cit., p. 159.

the Goldschmidt solution furnishes therefore an absolute minimum for all ordinary curves joining A_0 and A_1 in the region R.

Case II. $|A_0A_1| < y_0 + y_1$. We construct the ellipse whose foci are A_0 and A_1 and whose points P satisfy the equation

$$|A_0P| + |A_1P| = y_0 + y_1,$$

and denote by R_0 the closed region consisting of the ellipse and its interior.

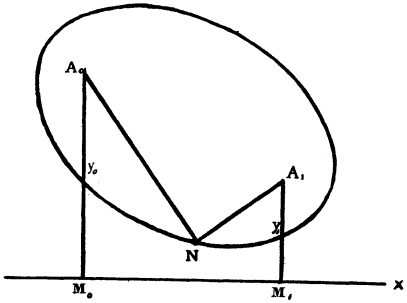

FIG. 3.

Then every point in the region R_0 is such that its ordinate y is greater than zero. For, if not, let N be one point of the ellipse, lying on the x-axis. Then $|A_0N| > y_0$ unless N coincides with M_0, and $|A_1N| > y_1$ unless N coincides with M_1, and therefore $|A_0N| + |A_1N| > y_0 + y_1$, since N cannot coincide with both M_0 and M_1. But this contradicts the hypothesis,

$$|A_0N| + |A_1N| = y_0 + y_1.$$

For the region R_0, the hypotheses of the Hilbert existence theorem* are fulfilled, since

$$\begin{aligned} &A) \quad F(x_1y, kx', ky') = y\sqrt{kx'^2 + k^2y'^2} = kF(xyx'y'), \\ &B) \quad F(xy \cos\gamma \sin\gamma) = y > 0, \\ &C) \quad F_1(xy \cos\gamma \sin\gamma) = y > 0, \end{aligned}$$

D) R_0 is convex and A_0 and A_1 are interior points.

Hence we may apply the Hilbert existence theorem to the region R_0, obtaining the result that among all rectifiable curves in R_0 joining A_0 and A_1 there exists at least one curve $\mathfrak{C}$ furnishing an absolute minimum for the integral I. This curve is either an extremal $\mathfrak{E}$ of class C', i. e., a catenary with x-axis as directrix, or consists of portions of extremals separated by points of the boundary R_0. In the latter case, its length would be greater than $y_0 + y_1$, for it could not consist merely of two straight lines from A_0 and A_1 to a single point P of the boundary, since these straight lines could not both be extremals. By the theorem proved above, such a curve would give a value of I greater than that given by the Goldschmidt solution.

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* Bolza, loc. cit., §43.

Among all ordinary curves not wholly in R_0, the Goldschmidt solution furnishes an absolute minimum, since every such curve contains at least one point of the ellipse and its length is greater than $y_0 + y_1$. We have therefore the theorem:

Among all ordinary curves in R, $(y \geqq 0)$, *joining* A_0 *and* A_1, *there exists a curve which furnishes for the integral* $I = \int y ds$ *an absolute minimum, and this curve is either a catenary through* A_0 *and* A_1, *not containing the conjugate point to* A_0, *if such exist, or it is the Goldschmidt discontinuous solution,* $A_0 M_0 M_1 A_1$.

Making use of the results of MacNeish* for relative minima, we now conclude:

1) *The catenary solution furnishes the absolute minimum if* A_1 *lies within the curve* F *defined for* A_0 $(x_0\, y_0)$ *by the equations*

$$F) \qquad \begin{aligned} x - x_0 &= a(t - t_0), \\ y = a \operatorname{ch} t, \; y_0 &= a \operatorname{ch} t_0, \\ 2t_0 + 1 + e^{2t_0} &= 2t - 1 - e^{-2t}. \end{aligned}$$

2) *Both solutions furnish the absolute minimum if* A_1 *lies on* F, *the values of* I *then being equal for the two solutions.*

3) *The Goldschmidt solution furnishes the absolute minimum if* A_1 *lies without* F.

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* MacNeish, ANNALS OF MATHEMATICS, vol. 7 (1906), p. 80.